Research Article

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On weakly 2-absorbing δ-primary ideals of commutative rings

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Abstract: Let R be a commutative ring with $1 \neq 0$. We recall that a proper ideal I of R is called a weakly 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. In this paper, we introduce a new class of ideals that is closely related to the class of weakly 2-absorbing primary ideals. Let $I(R)$ be the set of all ideals of R and let $\delta : I(R) \to I(R)$ be a function. Then δ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, then $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Let δ be an expansion function of ideals of R. Then a proper ideal I of R (i.e., $I \neq R$) is called a *weakly 2-absorbing δ-primary ideal* if $0 \neq abc \in I$ implies $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$. For example, let $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$. Then δ is an expansion function of ideals of R and hence a proper ideal I of R is a weakly 2-absorbing primary ideal of R if and only if I is a weakly 2-absorbing δ-primary ideal of R. A number of results concerning weakly 2-absorbing δ-primary ideals and examples of weakly 2-absorbing δ-primary ideals are given.

Keywords: Prime ideal, 2-absorbing ideal, weakly 2-absorbing ideal, weakly prime ideal, almost prime ideal, ϕ-prime ideal

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1 Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$. Let R be a commutative ring. An ideal I of R is said to be proper if $I \neq R$. Let I be a proper ideal of R. Then $\sqrt{I}$ denotes the radical ideal of I (i.e., $\sqrt{I} = \{x \in R \mid x^n \in I$ for some positive integer $n \geq 1\}$). Note that $\sqrt{0}$ is the set (ideal) of all nilpotent elements of R. The notion of 2-absorbing ideals, which is a generalization of prime ideals, was introduced by Badawi in [4] and studied by several authors (see, for instance, [2], [3], [11], and [16]). Some authors studied variations of the concept of a 2-absorbing ideal: for example, on 2-absorbing preradicals (see [9]), classical 2-absorbing submodules of modules over commutative rings (see [15]),and co-2-absorbing preradicals and submodules (see [10]).

Recall that a proper ideal I of R is called a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In [7], the concept of a 2-absorbing ideal was extended to the context of a 2-absorbing primary ideal which is a generalization of a primary ideal. Recall from [7] that a proper ideal of R is said to be a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Recall from [1] and [12] that a proper ideal I of R is called a weakly prime ideal (weakly primary ideal) if whenever $0 \neq ab \in I$, then $a \in I$ or $b \in I$ ($a \in I$ or $b \in \sqrt{I}$). The concept of a weakly prime ideal was extended to the context of a weakly 2-absorbing ideal. Recall from [6] that a proper ideal I of R is said to be a weakly 2-absorbing ideal of R if whenever $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The concept of a weakly primary ideal was extended to the context of a weakly 2-absorbing primary ideal. Recall from [8] that a proper ideal I of R is said to be a weakly 2-absorbing primary ideal

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of R if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Recently, the first-named author has extended the concept of weakly prime ideals to the context of weakly semi-prime ideals. Recall from [5] that a proper ideal I of R is called a weakly semi-prime ideal of R if $0 \neq a^2 \in I$ implies $a \in I$.

Let $I(R)$ be the set of all ideals of R. Dongsheng Zhao [17] introduced the concept of expansion of ideals of R. We recall from [17] that a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, then $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Recall from [17] that a proper ideal I of R is said to be a δ-primary ideal of R if $a, b \in R$ with $ab \in I$ implies $a \in I$ or $b \in \delta(I)$, where δ is an expansion function of ideals of R. The concept of δ-primary ideal was extended to the context of of 2-absorbing δ-primary ideal. Recall from [13] that a proper ideal I of R is said to be a 2-absorbing δ-primary ideal of R if $a, b, c \in R$ with $abc \in I$ implies $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$, where δ is an expansion function of ideals of R. Let δ be an expansion function of ideals of R and $I \in I(R)$. In this paper, we call I a *weakly 2-absorbing δ-primary ideal* if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$. Note that if $\delta(I) = I$ for every $I \in I(R)$, then a proper ideal I of R is a weakly 2-absorbing δ-primary ideal of R if and only if I is a weakly 2-absorbing ideal of R. Also, note that if $\delta(I) = \sqrt{I}$ for every $I \in I(R)$, then a proper ideal I of R is a weakly 2-absorbing δ-primary ideal of R if and only if I is a weakly 2-absorbing primary ideal of R.

Among the results in this paper, it is shown (Theorem 2.15) that if δ and γ are expansion functions of ideals of R and I is a proper ideal of R such that $\gamma(I)$ is a weakly δ-primary ideal of R, then I is a weakly 2-absorbing $\delta o \gamma$-primary ideal of R. It is shown (Theorem 2.9) that if δ is expansion function of ideals of R and I is a weakly 2-absorbing δ-primary ideal that is not 2-absorbing δ-primary, then $I^3 = 0$. It is shown (Example 3) that if $I^3 = 0$ for some proper ideal I of R, then I need not be a weakly 2-absorbing δ-primary ideal of R, for some expansion function δ of ideals of R. It is shown (Theorem 2.11) that if δ is an expansion function of ideals of R such that $\delta(0)$ is δ-primary and $\delta(\delta(0)) = \delta(0)$ and I is a weakly 2-absorbing δ-primary ideal of R that is not 2-absorbing δ-primary, then $\delta(I) = \delta(0)$ is a prime ideal of R. It is shown (Theorem 3.5) that if I is a weakly 2-absorbing δ-primary ideal of R and $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R such that I is a free δ-triple-zero with respect to $I_1I_2I_3$, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \delta(I)$ or $I_1I_3 \subseteq \delta(I)$. In the last section, some results concerning weakly 2-absorbing δ-primary ideals in the product of rings are given.

2 Properties of weakly 2-absorbing δ-primary ideals

We start by recalling the definition of a δ-function.

Definition 2.1. Let $I(R)$ be the set of all ideals of R. We recall from [17] that a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, then $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$.

In the following example, we give some expansion functions of ideals of a ring R.

Example 1. Let $\delta : I(R) \to I(R)$ be a function. Then

(1) If $\delta(I) = I$ for every ideal I of R, then δ is an expansion function of ideals of R, such a function is denoted by δ_I.

(2) If $\delta(I) = \sqrt{I}$ (note that $\sqrt{R} = R$) for every ideal I of R, then δ is an expansion function of ideals of R, such a function is denoted by $\delta_{\sqrt{I}}$.

(3) Suppose that R is a quasi-local ring with a maximal ideal M. If $\delta(I) = M$ for every proper ideal I of R and $\delta(R) = R$, then δ is an expansion function of ideals of R, such a function is denoted by δ_{Max}.

(4) Let I be a proper ideal of R. Recall from [14] that an element $r \in R$ is called an integral over I if there is an integer $n \geq 1$ and $a_i \in I^i$, $i = 1, \ldots, n$, $r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_{n-1}r + a_n = 0$. Let $\overline{I} = \{r \in R | r \text{ be integral over } I\}$. Let $I \in I(R)$. It is known (see [14]) that $\overline{I}$ is an ideal of R and $I \subseteq \overline{I} \subseteq \sqrt{I}$ and if $J \subseteq I$, then $\overline{J} \subseteq \overline{I}$. If $\delta(I) = \overline{I}$ for every ideal I of R, then δ is an expansion function of ideals of R, such a function is denoted by $\delta_{\overline{I}}$.

(5) Let J be a proper ideal of R. If $\delta(I) = I + J$ for every ideal I of R, then δ is an expansion function of ideals of R, such a function is denoted by δ_+.
(6) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : I(R) \to I(R)$ such that $\delta(I) = \delta_1(I) + \delta_2(I)$. Then δ is an expansion function of ideals of R, such a function is denoted by $\delta_\oplus$.
(7) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : I(R) \to I(R)$ such that $\delta(I) = \delta_1(I) \cap \delta_2(I)$. Then δ is an expansion function of ideals of R, such a function is denoted by $\delta_\cap$.
(8) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : I(R) \to I(R)$ such that $\delta(I) = (\delta_1 o \delta_2)(I) = \delta_1(\delta_2(I))$. Then δ is an expansion function of ideals of R, such a function is denoted by δ_o.

Remark 2.2. Note that $\delta_I(J) \subseteq \delta_{\overline{I}}(J) \subseteq \delta_{\sqrt{I}}(J)$ for every $J \in I(R)$.

We recall the following definitions.

Definition 2.3. Let δ be an expansion function of ideals of a ring R.
(1) A proper ideal I of R is called a *weakly 2-absorbing δ-primary ideal of R* if whenever $a, b, c \in R$, $0 \neq abc \in I$, then $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$.
(2) A proper ideal I of R is called a *weakly δ-primary ideal of R* if whenever $a, b \in R$, $0 \neq ab \in I$, then $a \in I$ or $b \in \delta(I)$.
(3) ([13]). A proper ideal I of R is called a 2-absorbing δ-primary ideal of R if whenever $a, b, c \in R$, $abc \in I$, then $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$.
(4) ([17]). A proper ideal I of R is called a δ-primary ideal of R if whenever $a, b \in R$, $ab \in I$, then $a \in I$ or $b \in \delta(I)$.

We begin with the following trivial result.

Theorem 2.4. *Let δ be an expansion function of ideals of R and $I \in I(R)$. Then*
(1) *If I is a 2-absorbing δ-primary ideal of R, then I is a weakly 2-absorbing δ-primary ideal of R.*
(2) *If I is a δ-primary ideal of R, then I is a weakly δ-primary ideal of R.*
(3) *If I is a weakly δ-primary ideal of R, then I is a weakly 2-absorbing δ-primary ideal of R.*
(4) *If I is a δ-primary ideal of R, then I is a 2-absorbing δ-primary ideal of R.*
(5) *Let δ_1 be an expansion function of ideal of R such that $\delta(I) \subseteq \delta_1(I)$. If I is a weakly 2-absorbing δ-primary ideal of R, then I is a weakly 2-absorbing δ_1-primary ideal of R.*
(6) *Let δ_1 be an expansion function of the ideal of R such that $\delta(I) \subseteq \delta_1(I)$. If I is a weakly δ-primary ideal of R, then I is a weakly δ_1-primary ideal of R.*

Proof. (1) & (2) The claim is clear by definition.

(3) Suppose that I is a weakly δ-primary ideal of R. Let $0 \neq abc \in I$ for some $a, b, c \in R$, and assume that $ab \notin I$. Since I is a weakly δ-primary ideal of R and $ab \notin I$, we have $c \in \delta(I)$. Thus $ac \in \delta(I)$. Hence I is a weakly 2-absorbing δ-primary ideal of R.

(4) By a similar argument as in the proof of (3), the claim follows.

(5) Since $\delta(I) \subseteq \delta_1(I)$ and I is a weakly 2-absorbing δ-primary ideal of R, the claim follows.

(6) Since $\delta(I) \subseteq \delta_1(I)$ and I is a weakly δ-primary ideal of R, the claim follows. ☐

The following is an example of a weakly 2-absorbing δ-primary ideal of R, for some expansion function δ of the ideal of R, that is neither a weakly 2-absorbing primary ideal of R nor a 2-absorbing δ-primary of R nor a weakly δ-primary ideal of R.

Example 2. Let $A = \mathbb{Z}_8[X, Y, W]$ and $I = X^2Y^2W^2A$ be an ideal of A. Let $R = \frac{A}{I}$ and define $\delta : I(R) \to I(R)$ such that $\delta(L) = L + \frac{xA+I}{I}$. Then it is clear that δ is an expansion function of ideals of R. Let $J = \frac{XYWA}{I}$. Then $\sqrt{J} = \frac{2A+XYWA}{I}$. We show that J is not a weakly 2-absorbing primary ideal of R. For the ring R, we have $0 \neq XYW + I \in J$, but $XY + I \notin J$, $XW + I \notin \sqrt{J}$, and $YW + I \notin \sqrt{J}$. Thus J is not a weakly 2-absorbing primary ideal of R (note that J is not a weakly 2-absorbing ideal of R). We show that J is not a 2-absorbing δ-primary of R. Let $x = 2 + I \in R$. Then $x^3 = 0 + I \in J$, but nether $x^2 = 4 + I \in J$ nor $x^2 = 4 + I \in \delta(I)$. Hence J is not a 2-absorbing δ-primary of R. We show that J is not a weakly δ-primary ideal of R, for $0 \neq XYW + I \in J$, but $XY + I \notin J$ and $W + I \notin \delta(J)$. Thus J is not a weakly δ-primary ideal of R. By the construction of δ, it is easily verified that J is a weakly 2-absorbing δ-primary ideal of R.

Theorem 2.5. *Let δ be an expansion function of ideals of R and I be a proper ideal of R. If $\delta(I)$ is a weakly prime ideal of R, then I is a weakly 2-absorbing δ-primary ideal of R.*

Proof. Suppose that $\delta(I)$ is a weakly prime ideal of R. Assume that $0 \neq abc \in I$ and $ab \notin I$ for some $a, b, c \in R$. We consider two cases. **Case one**: Suppose that $ab \notin \delta(I)$. Since $\delta(I)$ is weakly prime and $ab \notin \delta(I)$, we have $c \in \delta(I)$. Thus $ac \in \delta(I)$. **Case two**: Suppose that $ab \in \delta(I)$. Since $0 \neq abc \in I$, we have $0 \neq ab \in \delta(I)$. Since $\delta(I)$ is a weakly prime ideal of R, we have $a \in \delta(I)$ or $b \in \delta(I)$. Thus $ac \in \delta(I)$ or $bc \in \delta(I)$. Thus I is a weakly 2-absorbing δ-primary ideal of R. □

Theorem 2.6. *Let δ be an expansion function of ideals of R and $L \subseteq I$ be proper ideals of R. Suppose that I is a weakly δ-primary ideal of R such that $\delta(L) = \delta(I)$. Let J be an ideal of R such that $L \subseteq J \subseteq I$. Then J is a weakly 2-absorbing δ-primary ideal of R.*

Proof. Since $L \subseteq J \subseteq I$ and $\delta(L) = \delta(I)$, we have $\delta(J) = \delta(I)$. Let $0 \neq abc \in J$ for some $a, b, c \in R$ and suppose that $ab \notin J$. Since $J \subseteq I$, we have $0 \neq abc \in I$. We consider two cases. **Case one**: Suppose that $ab \notin I$. Since I is weakly δ-primary and $ab \notin I$, we have $c \in \delta(I) = \delta(J)$. Thus $ac \in \delta(J)$. **Case two**: Suppose that $ab \in I$. Since $0 \neq abc \in I$, we have $0 \neq ab \in I$. Since I is a weakly δ-primary ideal of R, we have $a \in I \subseteq \delta(J)$ or $b \in \delta(J)$. Thus $ac \in \delta(J)$ or $bc \in \delta(J)$. Thus J is a weakly 2-absorbing δ-primary ideal of R. □

Definition 2.7. Let δ be an expansion function of ideals of R.
(1) Suppose that I is a weakly δ-primary ideal of R. We say (a, b) is a δ-twin-zero of I if $ab = 0$, $a \notin I$, and $b \notin \delta(I)$.
(2) Suppose that I is a weakly 2-absorbing δ-primary ideal of R. We say (a, b, c) is a δ-triple-zero of I if $abc = 0$, $ab \notin I$, $bc \notin \delta(I)$, and $ac \notin \delta(I)$.

Theorem 2.8. *Let δ be an expansion function of ideals of R. Then*
(1) *Let I be a weakly δ-primary ideal of R and suppose that (a, b) is a δ-twin-zero of I for some $a, b \in R$. Then $aI = bI = 0$.*
(2) *Let I be a weakly 2-absorbing δ-primary ideal of R and suppose that (a, b, c) is a δ-triple-zero of I for some $a, b, c \in R$. Then*
  (i) $abI = bcI = acI = 0$.
  (ii) $aI^2 = bI^2 = cI^2 = 0$.

Proof. (1) Suppose that $aI \neq 0$. Then there exists $i \in I$ such that $ai \neq 0$. Hence $a(b + i) \neq 0$. Since $a \notin I$ and I is weakly δ-primary, we have $b + i \in \delta(I)$, and hence $b \in \delta(I)$, a contradiction. Thus $aI = 0$. Similarly, it can be easily verified that $bI = 0$.
(2) (i) Suppose that $abI \neq 0$. Then there exists $i \in I$ such that $abi \neq 0$. Hence $ab(c + i) \neq 0$. Since $ab \notin I$ and I is weakly 2-absorbing δ-primary, we have $a(c + i) \in \delta(I)$ or $b(c + i) \in \delta(I)$. So $ac \in \delta(I)$ or $bc \in \delta(I)$, a contradiction. Thus $abI = 0$. Similarly it can be easily verified that $bcI = acI = 0$.
  (ii) Suppose that $ai_1i_2 \neq 0$ for some $i_1, i_2 \in I$. Hence from (1) we have $a(b + i_1)(c + i_2) = ai_1i_2 \neq 0$. It implies either $a(b + i_1) \in I$ or $a(c + i_2) \in \delta(I)$ or $(b + i_1)(c + i_2) \in \delta(I)$. Thus $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$, a contradiction. Therefore $aI^2 = 0$. Similarly, one can easily show that $bI^2 = cI^2 = 0$. □

Theorem 2.9. *Let δ be an expansion function of ideals of R. Then*
(1) *If I is a weakly δ-primary ideal that is not δ-primary, then $I^2 = 0$.*
(2) *If I is a weakly 2-absorbing δ-primary ideal of R, i.e., not 2-absorbing δ-primary, then $I^3 = 0$.*

Proof. (1) Let (a,b) be a δ-twin-zero of I. Suppose that $i_1i_2 \neq 0$ for some $i_1, i_2 \in I$. Then by Theorem 2.8(1), we have $(a+i_1)(b+i_2) = i_1i_2 \neq 0$. Thus $(a+i_1) \in I$ or $(b+i_2) \in \delta(I)$, and hence $a \in I$ or $b \in \delta(I)$, a contradiction. Therefore $I^2 = 0$.

(2) Suppose that I is a weakly 2-absorbing δ-primary ideal that is not a 2-absorbing δ-primary ideal of R. Then there exists (a,b,c) a δ-triple-zero of I for some $a,b,c \in R$. Assume that $I^3 \neq 0$. Hence $i_1i_2i_3 \neq 0$, for some $i_1, i_2, i_3 \in I$. By Theorem 2.8, we obtain $(a+i_1)(b+i_2)(c+i_3) = i_1i_2i_3 \neq 0$. This implies that $(a+i_1)(b+i_2) \in I$ or $(a+i_1)(c+i_3) \in \delta(I)$ or $(b+i_2)(c+i_3) \in \delta(I)$. Thus we have $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$, a contradiction. Thus $I^3 = 0$. □

The following example shows that a proper ideal I of R with the property $I^3 = 0$ need not be a weakly 2-absorbing $\delta_{\sqrt{I}}$-primary ideal of R. We have the following example.

Example 3. Let $R = \mathbb{Z}_{90}$. Then $I = \{0, 30, 60\}$ is an ideal of R. Clearly, $I^3 = 0$ and $0 \neq 2 \times 3 \times 5 = 30 \in I$. Since $2 \times 3 = 6 \notin I$, $2 \times 5 = 10 \notin \delta_{\sqrt{I}}(I)$, and $3 \times 5 = 15 \notin \delta_{\sqrt{I}}(I)$, we conclude that I is not a weakly 2-absorbing $\delta_{\sqrt{I}}$-primary ideal of R.

Theorem 2.10. *Let I be a proper ideal of a ring R. Then I is a weakly 2-absorbing $\delta_{\overline{I}}$ -primary ideal of R that is not 2-absorbing $\delta_{\overline{I}}$ -primary if and only if I is a weakly 2-absorbing primary ideal of R that is not 2-absorbing primary.*

Proof. Suppose that I is a weakly 2-absorbing $\delta_{\overline{I}}$ -primary ideal of R that is not 2-absorbing $\delta_{\overline{I}}$ -primary. Then $I^3 = 0$ by Theorem 2.9(2). Thus $I \subseteq \sqrt{0} = \sqrt{I}$. Since $\sqrt{0} \subseteq \overline{I} \subseteq \sqrt{I} = \sqrt{0}$, we conclude that $\overline{I} = \sqrt{0}$ and hence I is a weakly 2-absorbing primary ideal of R that is not 2-absorbing primary.

Conversely, I is a weakly 2-absorbing primary ideal of R that is not 2-absorbing primary. Hence I is a weakly 2-absorbing $\delta_{\sqrt{I}}$-primary ideal of R that is not 2-absorbing $\delta_{\sqrt{I}}$ -primary. Thus $I^3 = 0$, and hence $\sqrt{I} = \sqrt{0}$. Since $\sqrt{0} \subseteq \overline{I} \subseteq \sqrt{I} = \sqrt{0}$, we conclude that $\overline{I} = \sqrt{0}$ and hence I is a weakly 2-absorbing $\delta_{\overline{I}}$ -primary ideal of R that is not 2-absorbing $\delta_{\overline{I}}$ -primary. □

Theorem 2.11. *Let δ be an expansion function of ideals of R such that $\delta(0)$ is a δ-primary ideal of R and $\delta(\delta(0)) = \delta(0)$. Then*

(1) *$\delta(0)$ is a prime ideal of R.*

(2) *If I is a weakly δ-primary ideal that is not δ-primary, then $\delta(I) = \delta(0)$ is a prime ideal of R.*

(3) *If I is a weakly 2-absorbing δ-primary ideal of R that is not 2-absorbing δ-primary, then $\delta(I) = \delta(0)$.*

Proof. (1) Suppose that $ab \in \delta(0)$ and $a \notin \delta(0)$ for some $a, b \in R$. Since $\delta(0)$ is a δ-primary ideal of R and $a \notin \delta(0)$, we have $b \in \delta(\delta(0)) = \delta(0)$. Thus $\delta(0)$ is a prime ideal of R.

(2) Clearly, $\delta(0) \subseteq \delta(I)$. Since $I^2 = 0$ by Theorem 2.9 and $\delta(0)$ is a prime ideal of R, we have $I \subseteq \delta(0)$. Since $\delta(\delta(0)) = \delta(0)$, we have $\delta(I) \subseteq \delta(\delta(0)) = \delta(0)$. Since $\delta(0) \subseteq \delta(I)$ and $\delta(I) \subseteq \delta(0)$, we have $\delta(I) = \delta(0)$ is a prime ideal of R.

(3) Again, it is clear that $\delta(0) \subseteq \delta(I)$. Since $I^3 = 0$ by Theorem 2.9 and $\delta(0)$ is a prime ideal of R, we have $I \subseteq \delta(0)$. Since $\delta(\delta(0)) = \delta(0)$, we have $\delta(I) \subseteq \delta(\delta(0)) = \delta(0)$. Since $\delta(0) \subseteq \delta(I)$ and $\delta(I) \subseteq \delta(0)$, we have $\delta(I) = \delta(0)$ is a prime ideal of R. □

Theorem 2.12. *Let δ be an expansion function of ideals of R such that $\delta(0)$ is a δ-primary ideal of R and $\delta(\delta(0)) = \delta(0)$. Let I be a weakly δ-primary ideal of R that is not a δ-primary ideal of R and J be an ideal of R such that $J \subseteq I$. Then J is a weakly 2-absorbing δ-primary ideal of R. In particular, if L is an ideal of R, then $A = I \cap L$ and $B = IL$ are weakly 2-absorbing δ-primary ideals of R.*

Proof. Since I is a weakly δ-primary ideal of R that is not δ-primary, $\delta(I) = \delta(0)$ by Theorem 2.11. Hence $\delta(J) = \delta(I) = \delta(0)$. Let $0 \neq abc \in J$ for some $a, b, c \in R$ and suppose that $ab \notin J$. Since $J \subseteq I$, we have $0 \neq abc \in I$. We consider two cases. **Case one**: Suppose that $ab \notin I$. Since I is weakly δ-primary and $ab \notin I$, we have $c \in \delta(J) = \delta(I) = \delta(0)$. Thus $ac \in \delta(J)$. **Case two**: Suppose that $ab \in I$. Since $0 \neq abc \in I$, we have $0 \neq ab \in I$. Since I is a weakly δ-primary ideal of R, we have $a \in I \subseteq \delta(0)$ or $b \in \delta(0)$. Thus $ac \in \delta(J)$ or $bc \in \delta(J)$. Thus J is a weakly 2-absorbing δ-primary ideal of R. The proof of the “in particular statement” is clear since $A, B \subseteq I$. □

Recall that a ring R is said to be reduced if $\sqrt{0} = 0$. A ring R is said to be δ-*reduced* if $\delta(0) = 0$.

Theorem 2.13. *Let δ be an expansion function of ideals of R such that $\delta(0)$ is a δ-primary ideal of R and suppose that R is δ-reduced. Then*
(1) *A proper ideal I is a weakly δ-primary ideal of R if and only if I is a δ-primary ideal of R.*
(2) *A proper ideal I is a weakly 2-absorbing δ-primary ideal if and only if I is a 2-absorbing δ-primary ideal of R.*

Proof. Since R is δ-reduced, we have $\delta(0) = 0$ and hence $\delta(\delta(0)) = \delta(0) = 0$. Since $\delta(0) = 0$ is a δ-primary ideal of R, we conclude that $\delta(0) = 0$ is a prime ideal of R by Theorem 2.11.
(1) Suppose that $I \neq 0$ and I is a weakly δ-primary ideal of R. If I is not a δ-primary ideal of R, then $\delta(I) = \delta(0) = 0$ by Theorem 2.11 and thus $I = 0$, a contradiction.
(2) Suppose $I \neq 0$ and I is a weakly 2-absorbing δ-primary ideal of R. If I is not a 2-absorbing δ-primary ideal of R, then $\delta(I) = \delta(0) = 0$ by Theorem 2.11 and thus $I = 0$, a contradiction. □

Theorem 2.14. *Let δ, γ be expansion functions of ideals of R such that $\delta(0)$ is a γ-primary ideal of R. Let I be a proper ideal of R. Then I is a weakly 2-absorbing $\gamma o \delta$-primary ideal of R if and only if I is a 2-absorbing $\gamma o \delta$-primary ideal of R.*

Proof. Suppose that I is a weakly 2-absorbing $\gamma o\delta$-primary ideal of R. Assume that $abc \in I$ for some $a, b, c \in R$. If $0 \neq abc \in I$, then $ab \in I$ or $ac \in \gamma(\delta(I))$ or $bc \in \gamma(\delta(I))$. Hence assume that $abc = 0$ and $ab \notin I$. Since $abc = 0$ and $\delta(0)$ is a γ-primary ideal of R, we conclude that $a \in \delta(0)$ or $b \in \gamma(\delta(0))$ or $c \in \gamma(\delta(0))$. Since $\gamma(\delta(0)) \subseteq \gamma(\delta(I))$, we conclude that $ac \in \gamma(\delta(0)) \subseteq \gamma(\delta(I))$ or $bc \in \gamma(\delta(0)) \subseteq \gamma(\delta(I))$. Thus I is a 2-absorbing $\gamma o\delta$-primary ideal of R. The converse is clear. □

Theorem 2.15. *Let δ and γ be expansion functions of ideals of R, and let I be a proper ideal of R such that $\gamma(I)$ is a weakly δ-primary ideal of R. Then I is a weakly 2-absorbing $\delta o\gamma$-primary ideal of R.*

Proof. Suppose that $0 \neq abc \in I$ for some $a, b, c \in R$ and $ab \notin I$. Suppose that $ab \notin \gamma(I)$. Since $\gamma(I)$ is a weakly δ-primary ideal of R, we have $c \in \delta(\gamma(I))$, and thus $ac \in \delta(\gamma(I))$. Suppose that $ab \in \gamma(I)$. Since $0 \neq abc \in I$ and $ab \in \gamma(I)$, we have $0 \neq ab \in \gamma(I)$. Since $\gamma(I)$ is a weakly δ-primary ideal of R and $0 \neq ab \in \gamma(I)$, we have $a \in \delta(\gamma(I))$ or $b \in \delta(\gamma(I))$. Thus $ac \in \delta(\gamma(I))$ or $bc \in \delta(\gamma(I))$. Thus I is a weakly 2-absorbing $\delta o\gamma$-primary ideal of R. □

Recall from [17] that an expansion function δ of ideals of R is intersection preserving if $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for every $I, J \in I(R)$.

Theorem 2.16. *Let δ be an expansion function of ideals of R such that $\delta(0)$ is δ-primary, $\delta(\delta(0)) = \delta(0)$ and δ is intersection preserving. Let $I_1, I_2, \ldots, I_n$ be weakly 2-absorbing δ-primary ideals of R such that every I_i, $i = 1, \ldots, n$, is not 2-absorbing δ-primary. Then $I = \cap_{i=1}^{n} I_i$ is a weakly 2-absorbing δ-primary ideal of R.*

Proof. Observe that $\delta(I_i) = \delta(0)$ for each $1 \leq i \leq n$ by Theorem 2.11. Since δ is intersection preserving, we conclude that $\delta(I) = \delta(0)$. Suppose that $a, b, c \in R$ with $0 \neq abc \in I$ and $ab \notin I$. Then $ab \notin I_k$ for some $1 \leq k \leq n$. Hence $bc \in \delta(I_k) = \delta(0) = \delta(I)$ or $ac \in \delta(I_k) = \delta(0) = \delta(I)$. Hence I is a weakly δ-2-absorbing ideal of R. □

Let $f : R \longrightarrow S$ be a ring-homomorphism, γ be an expansion function of ideals of R, and δ be an expansion function of ideals of S. We say that f is a $\gamma\delta$-*ring-homomorphism* if $\gamma(f^{-1}(I)) = f^{-1}(\delta(I))$ for all $I \in I(S)$. Note that if f is a surjective $\gamma\delta$-ring-homomorphism and $ker(f) \subseteq I$, for some ideal $I \in I(R)$, then $f(\gamma(I)) = \delta(f(I))$. In particular, if f is a $\gamma\delta$-ring-isomorphism, then $f(\gamma(I)) = \delta(f(I))$ for every ideal I of R.

Theorem 2.17. *Let $f : R \longrightarrow R'$ be a ring-homomorphism, γ be an expansion function of ideals of R, and δ be an expansion function of ideals of R'. Suppose that f is a $\gamma\delta$-ring-homomorphism. Then*

(1) *If f is a monomorphism and J' is a weakly 2-absorbing δ-primary ideal of R', then $f^{-1}(J')$ is a weakly 2-absorbing γ-primary ideal of R.*
(2) *If f is an epimorphism and J is a weakly 2-absorbing γ-primary ideal of R containing $Ker(f)$, then $f(J)$ is a weakly 2-absorbing δ-primary ideal of R'.*

Proof. (1) Let $a, b, c \in R$ such that $0 \neq abc \in f^{-1}(J')$. Since $Ker(f) = 0$, we get $0 \neq f(abc) = f(a)f(b)f(c) \in J'$. Hence we have $f(a)f(b) \in J'$ or $f(b)f(c) \in \delta(J')$ or $f(a)f(c) \in \delta(J')$, and thus $ab \in f^{-1}(J')$ or $bc \in f^{-1}(\delta(J'))$ or $ac \in f^{-1}(\delta(J'))$. Since $f^{-1}(\delta(J')) = \gamma(f^{-1}(J'))$, we conclude that $f^{-1}(J')$ is a weakly 2-absorbing γ-primary ideal of R.
(2) Let $a', b', c' \in R'$ and $0 \neq a'b'c' \in f(J)$. Then there exist $a, b, c \in R$ such that $f(a) = a', f(b) = b', f(c) = c'$ and $0 \neq f(abc) = a'b'c' \in f(J)$. Since $Ker(f) \subseteq J$, we have $0 \neq abc \in J$. It implies that $ab \in J$ or $ac \in \gamma(J)$ or $bc \in \gamma(J)$. This means that $a'b' \in f(J)$ or $a'c' \in f(\gamma(J)) \subseteq \delta(f(J))$ or $b'c' \in f(\gamma(J)) \subseteq \delta(f(J))$. Thus $f(J)$ is a weakly 2-absorbing δ-primary ideal of R'. □

Theorem 2.18. *Let γ be an expansion function of ideals of R and let I, J be proper ideals of R with $I \subseteq J$. Let $\delta : I(\frac{R}{I}) \to I(\frac{R}{I})$ be an expansion function of ideals of $S = \frac{R}{I}$ such that $\delta(\frac{L+I}{I}) = \frac{\gamma(L+I)}{I}$ for every $L \in I(R)$. Then the followings statements hold.*
(1) *Let $f : R \to S = \frac{R}{I}$ such that $f(r) = r + I$ for every $r \in R$. Then f is a surjective $\gamma\delta$-ring-homomorphism.*
(2) *If J is a weakly 2-absorbing γ-primary ideal of R, then J/I is a weakly 2-absorbing δ-primary ideal of S.*
(3) *If I is a 2-absorbing γ-primary ideal of R and J/I is a weakly 2-absorbing δ-primary ideal of S, then J is a 2-absorbing γ-primary ideal of R.*
(4) *If I is a weakly 2-absorbing δ-primary ideal of R and J/I is a weakly 2-absorbing δ-primary ideal of S, then J is a weakly 2-absorbing δ-primary ideal of R.*

Proof. (1) It is clear that f is surjective. Let $K \in I(S)$. Then $K = L + I$ for some $L \in I(R)$. Hence $f^{-1}(\delta(K)) = f^{-1}(\frac{\gamma(L+I)}{I}) = \gamma(L+I) = \gamma(f^{-1}(\frac{L+I}{I}))$. Thus f is a a surjective $\gamma\delta$-ring-homomorphism.
(2) It is obtained from Theorem 2.17(2).
(3) Let $a, b, c \in R$ and $abc \in J$. If $abc \in I$, then $ab \in I \subseteq J$ or $bc \in \gamma(I) \subseteq \gamma(J)$ or $ac \in \gamma(I) \subseteq \gamma(J)$. So we may assume that $abc \notin I$. Then we have $I \neq (a+I)(b+I)(c+I) \in J/I$. Since J/I is a weakly 2-absorbing δ-primary ideal of R/I, we conclude $(a+I)(b+I) = ab+I \in J/I$ or $(a+I)(c+I) = ac+I \in \delta(J/I)$ or $(b+I)(c+I) = bc+I \in \delta(J/I)$. It follows that $ab \in J$ or $ac \in \gamma(J)$ or $bc \in \gamma(J)$. Thus J is a 2-absorbing γ-primary ideal of R.
(4) Let $a, b, c \in R$ and $0 \neq abc \in J$. Then by a similar argument as in (3), J is a weakly 2-absorbing γ-primary ideal of R. □

Theorem 2.19. *Let S be a multiplicatively closed subset of R such that $0 \notin S$, δ be an expansion function of ideals of R, and δ_S be an expansion function of ideals of R_S such that $\delta_S(I_S) = \delta(I)_S$ for every $I \in I(R)$. If I is a weakly 2-absorbing δ-primary ideal of R with $I \cap S = \varnothing$, then I_S is a weakly 2-absorbing δ_S-primary ideal of R_S.*

Proof. Let $a, b, c \in R$, $s, t, k \in S$ such that $0 \neq \frac{a}{s}\frac{b}{t}\frac{c}{k} \in I_S$. Then there exists $u \in S$ such that $0 \neq uabc \in I$. Since I is a weakly 2-absorbing δ-primary ideal of R, we get either $uab \in I$ or $bc \in \delta(I)$ or $uac \in \delta(I)$. If $uab \in I$, then $\frac{a}{s}\frac{b}{t} = \frac{uab}{ust} \in I_S$. If $bc \in \delta(I)$, then $\frac{b}{t}\frac{c}{k} \in \delta(I)_S = \delta_S(I_S)$. If $uac \in \delta(I)$, then $\frac{a}{s}\frac{c}{k} = \frac{uac}{usk} \in \delta(I)_S = \delta_S(I_S)$. Thus I_S is a weakly 2-absorbing δ_S-primary ideal of R_S. □

We terminate this section with the following result.

Theorem 2.20. *Let δ be an expansion function of ideals of R and let I, J be proper ideals of R. Suppose that I, J are δ-primary ideals of R such that $\delta(I \cap J) = \delta(I) \cap \delta(J)$. Then $L = I \cap J$ is a 2-absorbing δ-primary ideal of R.*

Proof. Suppose that $abc \in L$ for some $a, b, c \in R$ and assume that $ab \notin L$. We consider three cases. **Case one**: Suppose that $ab \in I$ and $ab \notin J$. Since J is a δ-primary ideal of R, we have $c \in \delta(J)$.

Since I is a δ-primary ideal of R, we have $a \in \delta(I)$ or $b \in \delta(I)$. Thus $ac \in \delta(I \cap J) = \delta(I) \cap \delta(J)$ or $bc \in \delta(I \cap J) = \delta(I) \cap \delta(J)$. **Case two**: Suppose that $ab \notin I$ and $ab \in J$. By a similar argument as in case I, we conclude that $ac \in \delta(I \cap J) = \delta(I) \cap \delta(J)$ or $bc \in \delta(I \cap J) = \delta(I) \cap \delta(J)$. **Case three**: Suppose that $ab \notin I$ and $ab \notin J$. Since I, J are δ-primary ideals of R, we conclude that $c \in \delta(I) \cap \delta(J)$. Thus $ac, bc \in \delta(I \cap J) = \delta(I) \cap \delta(J)$. Hence $L = I \cap J$ is a 2-absorbing δ-primary ideal of R. □

By a similar argument as in the proof of Theorem 2.20, one can prove the following result.

Theorem 2.21. *Let δ be an expansion function of ideals of R and let I, J be proper ideals of R. Suppose that I, J are weakly δ-primary ideals of R such that $\delta(I \cap J) = \delta(I) \cap \delta(J)$. Then $L = I \cap J$ is a weakly 2-absorbing δ-primary ideal of R.*

3 Strongly weakly 2-absorbing δ-primary ideals

Definition 3.1. Let δ be an expansion function of ideals of R and let I be a proper ideal of R. Then I is called a *strongly weakly 2-absorbing δ-primary ideal of* R if whenever $0 \neq I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R, then $I_1 I_2 \subseteq I$ or $I_2 I_3 \subseteq \delta(I)$ or $I_1 I_3 \subseteq \delta(I)$.

Definition 3.2. Let δ be an expansion function of ideals of R and assume that I is a weakly 2-absorbing δ-primary ideal of R. Suppose that $0 \neq I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R. We say I is a free δ-triple-zero with respect to $I_1 I_2 I_3$ if (a, b, c) is not a δ-triple-zero of I for every $a \in I_1, b \in I_2$, and $c \in I_3$. We say that I is a free δ-triple-zero if whenever $0 \neq I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R, then I is free δ-triple-zero with respect to $I_1 I_2 I_3$.

The following conjecture is an analogue of [8, Conjecture 1].

Conjecture 1. Let δ be an expansion function of ideals of R. If I is a weakly 2-absorbing δ-primary ideal of R and $0 \neq I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R, then I is a free δ-triple-zero with respect to $I_1 I_2 I_3$.

The following result is an analogue of [8, Lemma 2.29].

Lemma 3.3. *Let δ be an expansion function of ideals of R. Let I be a weakly 2-absorbing δ-primary ideal of a ring R and suppose that $abJ \subseteq I$ for some elements $a, b \in R$ and some ideal J of R such that (a, b, c) is not a δ-triple-zero of I for every $c \in J$. If $ab \notin I$, then $aJ \subseteq \delta(I)$ or $bJ \subseteq \delta(I)$.*

Proof. Suppose that $aJ \nsubseteq \delta(I)$ and $bJ \nsubseteq \delta(I)$. Then $aj_1 \notin \delta(I)$ and $bj_2 \notin \delta(I)$ for some $j_1, j_2 \in J$. Since (a, b, j_1) is not a δ-triple-zero of I and $abj_1 \in I$ and $ab \notin I$ and $aj_1 \notin \delta(I)$, we have $bj_1 \in \delta(I)$. Since (a, b, j_2) is not a δ-triple-zero of I and $abj_2 \in I$ and $ab \notin I$ and $bj_2 \notin \delta(I)$, we have $aj_2 \in \delta(I)$. Now, since $(a, b, j_1 + j_2)$ is not a δ-triple-zero of I and $ab(j_1 + j_2) \in I$ and $ab \notin I$, we have $a(j_1 + j_2) \in \delta(I)$ or $b(j_1 + j_2) \in \delta(I)$. Suppose that $a(j_1 + j_2) = aj_1 + aj_2 \in \delta(I)$. Since $aj_2 \in \delta(I)$, we have $aj_1 \in \delta(I)$, a contradiction. Suppose that $b(j_1 + j_2) = bj_1 + bj_2 \in \delta(I)$. Since $bj_1 \in \delta(I)$, we have $bj_2 \in \delta(I)$, a contradiction again. Thus $aJ \subseteq \delta(I)$ or $bJ \subseteq \delta(I)$. □

Remark 3.4. Let δ be an expansion function of ideals of R. If I be a weakly 2-absorbing δ-primary ideal of R and suppose that $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R such that I is free δ-triple-zero with respect to $I_1 I_2 I_3$. Then if $a \in I_1, b \in I_2$, and $c \in I_3$, then $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$.

Let δ be an expansion function of ideals of R. Let I be a weakly 2-absorbing δ-primary ideal of R. In view of the result below, one can see that Conjecture 1 is valid if and only if whenever $0 \neq I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R such that I is a free δ-triple-zero with respect to $I_1 I_2 I_3$. Then $I_1 I_2 \subseteq I$ or $I_2 I_3 \subseteq \delta(I)$ or $I_1 I_3 \subseteq \delta(I)$.

The following result is an analogue of [8, Theorem 2.30].

Theorem 3.5. *Let δ be an expansion function of ideals of R. If I is a weakly 2-absorbing δ-primary ideal of R and it is supposed that $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R such that I is a free δ-triple-zero with respect to $I_1I_2I_3$, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \delta(I)$ or $I_1I_3 \subseteq \delta(I)$.*

Proof. Suppose that I is a weakly 2-absorbing δ-primary ideal of R and $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R such that I is free triple-zero with respect to $I_1I_2I_3$. Suppose that $I_1I_2 \nsubseteq I$. By Remark 3.4, we proceed with the same argument as in the proof of [8, Theorem 2.30]. We show that $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \delta(I)$. Suppose that neither $I_1I_3 \subseteq \delta(I)$ nor $I_2I_3 \subseteq \delta(I)$. Then there are $q_1 \in I_1$ and $q_2 \in I_2$ such that neither $q_1I_3 \subseteq \delta(I)$ nor $q_2I_3 \subseteq \delta(I)$. Since $q_1q_2I_3 \subseteq I$ and neither $q_1I_3 \subseteq \delta(I)$ nor $q_2I_3 \subseteq \delta(I)$, we have $q_1q_2 \in I$ by Lemma 3.3.

Since $I_1I_2 \nsubseteq I$, we have $ab \notin I$ for some $a \in I_1$, $b \in I_2$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq \delta(I)$ or $bI_3 \subseteq \delta(I)$ by Lemma 3.3. We consider three cases.

Case one: Suppose that $aI_3 \subseteq \delta(I)$, but $bI_3 \nsubseteq \delta(I)$. Since $q_1bI_3 \subseteq I$ and neither $bI_3 \subseteq \delta(I)$ nor $q_1I_3 \subseteq \delta(I)$, we conclude that $q_1b \in I$ by Lemma 3.3. Since $(a+q_1)bI_3 \subseteq I$ and $aI_3 \subseteq \delta(I)$, but $q_1I_3 \nsubseteq \delta(I)$, we conclude that $(a+q_1)I_3 \nsubseteq \delta(I)$. Since neither $bI_3 \subseteq \delta(I)$ nor $(a+q_1)I_3 \subseteq \delta(I)$, we conclude that $(a+q_1)b \in I$ by Lemma 3.3. Since $(a+q_1)b = ab + q_1b \in I$ and $q_1b \in I$, we conclude that $ab \in I$, a contradiction.

Case two: Suppose that $bI_3 \subseteq \delta(I)$, but $aI_3 \nsubseteq \delta(I)$. Since $aq_2I_3 \subseteq I$ and neither $aI_3 \subseteq \delta(I)$ nor $q_2I_3 \subseteq \delta(I)$, we conclude that $aq_2 \in I$. Since $a(b+q_2)I_3 \subseteq I$ and $bI_3 \subseteq \delta(I)$, but $q_2I_3 \nsubseteq \delta(I)$, we conclude that $(b+q_2)I_3 \nsubseteq \delta(I)$. Since neither $aI_3 \subseteq \delta(I)$ nor $(b+q_2)I_3 \subseteq \delta(I)$, we conclude that $a(b+q_2) \in I$ by Lemma 3.3. Since $a(b+q_2) = ab + aq_2 \in I$ and $aq_2 \in I$, we conclude that $ab \in I$, a contradiction.

Case three: Suppose that $aI_3 \subseteq \delta(I)$ and $bI_3 \subseteq \delta(I)$. Since $bI_3 \subseteq \delta(I)$ and $q_2I_3 \nsubseteq \delta(I)$, we conclude that $(b+q_2)I_3 \nsubseteq \delta(I)$. Since $q_1(b+q_2)I_3 \subseteq I$ and neither $q_1I_3 \subseteq \delta(I)$ nor $(b+q_2)I_3 \subseteq \delta(I)$, we conclude that $q_1(b+q_2) = q_1b + q_1q_2 \in I$ by Lemma 3.3. Since $q_1q_2 \in I$ and $q_1b + q_1q_2 \in I$, we conclude that $bq_1 \in I$. Since $aI_3 \subseteq \delta(I)$ and $q_1I_3 \nsubseteq \delta(I)$, we conclude that $(a+q_1)I_3 \nsubseteq \delta(I)$. Since $(a+q_1)q_2I_3 \subseteq I$ and neither $q_2I_3 \subseteq \delta(I)$ nor $(a+q_1)I_3 \subseteq \delta(I)$, we conclude that $(a+q_1)q_2 = aq_2 + q_1q_2 \in I$ by Lemma 3.3. Since $q_1q_2 \in I$ and $aq_2 + q_1q_2 \in I$, we conclude that $aq_2 \in I$. Now, since $(a+q_1)(b+q_2)I_3 \subseteq I$ and neither $(a+q_1)I_3 \subseteq \delta(I)$ nor $(b+q_2)I_3 \subseteq \delta(I)$, we conclude that $(a+q_1)(b+q_2) = ab + aq_2 + bq_1 + q_1q_2 \in I$ by Lemma 3.3. Since $aq_2, bq_1, q_1q_2 \in I$, we have $aq_2 + bq_1 + q_1q_2 \in I$. Since $ab + aq_2 + bq_1 + q_1q_2 \in I$ and $aq_2 + bq_1 + q_1q_2 \in I$, we conclude that $ab \in I$, a contradiction. Hence $I_1I_3 \subseteq \delta(I)$ or $I_2I_3 \subseteq \delta(I)$. ☐

Let δ be an expansion function of ideals of R. It is clear that if a proper ideal I of R is a strongly weakly 2-absorbing δ-primary ideal of R, then I is a weakly 2-absorbing δ-primary ideal of R. Theorem 3.5 gives a partially affirmative answer to the following conjecture.

Conjecture 2. Let δ be an expansion function of ideals of R and let I be a proper ideal of R. Then the following statements are equivalent.
(1) I is a strongly weakly 2-absorbing δ-primary ideal of R.
(2) I is a weakly 2-absorbing δ-primary ideal of R.
(3) I is a free δ-triple-zero.

4 Weakly 2-absorbing δ-primary ideals in product of rings

Let $R_1, \ldots, R_n$, where $n \geq 2$, be commutative rings with $1 \neq 0$. Assume that $\delta_1, \ldots, \delta_n$ are expansion functions of ideals of $R_1, \ldots, R_n$, respectively. Let $R = R_1 \times \cdots \times R_n$. We define a function $\delta_\times : I(R) \to I(R)$ such that $\delta_\times(I_1 \times \cdots \times I_n) = \delta_1(I_1) \times \cdots \times \delta_n(I_n)$ for every $I_i \in I(R_i)$, where $1 \leq i \leq n$. Then it is clear that $\delta_\times$ is an expansion function of ideals of R. Note that every ideal of R is of the form $I_1 \times \cdots \times I_n$, where each I_i is an ideal of R_i, $1 \leq i \leq n$.

Theorem 4.1. *Let R_1 and R_2 be commutative rings with $1 \neq 0$, δ_1, δ_2 be expansion functions of ideals of R_1, R_2, respectively. Let I be a proper ideal of R_1, and $R = R_1 \times R_2$. Then the following statements are equivalent.*
(1) *$I \times R_2$ is a weakly 2-absorbing $\delta_\times$-primary ideal of R.*
(2) *$I \times R_2$ is a 2-absorbing $\delta_\times$-primary ideal of R.*
(3) *I is a 2-absorbing δ_1-primary ideal of R_1.*

Proof. (1)⇒(2) Let $J = I \times R_2$. Then $J^3 \neq (0, 0)$. Hence J is a 2-absorbing $\delta_\times$-primary ideal of R by Theorem 2.9(2).

(2)⇒(3) Suppose that I is not a 2-absorbing δ_1-primary ideal of R_1. Then there exist $a, b, c \in R_1$ such that $abc \in I$, but $ab \notin I$, $bc \notin \delta_1(I)$, and $ac \notin \delta_1(I)$. Since $(a, 1)(b, 1)(c, 1) \in I \times R_2$, we have $(a, 1)(b, 1) = (ab, 1) \in I \times R_2$ or $(a, 1)(c, 1) = (ac, 1) \in \delta_\times(I \times R_2)$ or $(b, 1)(c, 1) = (bc, 1) \in \delta_\times(I \times R_2)$. It follows that $ab \in I$ or $bc \in \delta_1(I)$ or $ac \in \delta_1(I)$, a contradiction. Thus I is a 2-absorbing δ_1-primary ideal of R_1.

(3)⇒(1) Let I be a 2-absorbing δ_1-primary ideal of R_1. Then it is clear that $I \times R_2$ is a 2-absorbing $\delta_\times$-primary ideal of R. □

Theorem 4.2. *Let R_1 and R_2 be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$, and δ_1, δ_2 be expansion functions of ideals of R_1, R_2, respectively. Let $J = I_1 \times I_2$ be a proper ideal of R, for some nonzero ideals I_1, I_2 of R_1, R_2, respectively, such that for every $i \in \{1, 2\}$, if $I_i \neq R_i$, then $\delta_i(I_i) \neq R_i$. Then the following statements are equivalent.*
(1) *$J = I_1 \times I_2$ is a weakly 2-absorbing $\delta_\times$-primary ideal of R.*
(2) *$I_1 = R_1$ and I_2 is a 2-absorbing δ_2-primary ideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing δ_1-primary ideal of R_1 or I_1 is a δ_1-primary ideal of R_1 and I_2 is a δ_2-primary ideal R_2.*
(3) *$I_1 \times I_2$ is a 2-absorbing $\delta_\times$-primary ideal of R.*

Proof. (1)⇒(2) Assume that $I_1 \times I_2$ is a weakly 2-absorbing $\delta_\times$-primary ideal of R. If $I_1 = R_1$ ($I_2 = R_2$), then I_2 is a 2-absorbing δ_2-primary ideal of R_2 (I_1 is a 2-absorbing δ_1-primary ideal of R_1) by Theorem 4.1. So we may assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Let $a, b \in R_2$ such that $ab \in I_2$. Let $0 \neq x \in I_1$. Then $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \in I_1 \times I_2$. Thus $(x, 1) \times (1, a) = (x, a) \in I_1 \times I_2$ or $(1, a)(1, b) = (1, ab) \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ or $(x, 1)(1, b) = (x, b) \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$. Since $\delta_1(I_1) \neq R_1$, we have $(1, a)(1, b) = (1, ab) \notin \delta_\times(I_1 \times I_2)$. Hence $(x, 1) \times (1, a) = (x, a) \in I_1 \times I_2$ or $(x, 1)(1, b) = (x, b) \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$. Thus we conclude that $a \in I_2$ or $b \in \delta_2(I_2)$. Thus I_2 is a δ_2-primary ideal of R_2. Similarly, it can be easily shown that I_1 is a δ_1-primary ideal of R_1.

(2)⇒(3) If $J = I_1 \times R_2$ for some 2-absorbing δ_1-primary ideal I_1 of R_1 or $J = R_1 \times I_2$ for some 2-absorbing δ_2-primary ideal I_2 of R_2, then J is a 2-absorbing $\delta_\times$-primary ideal of R by Theorem 4.1. Hence assume that $J = I_1 \times I_2$ for some δ_1-primary ideal I_1 of R_1 and some δ_2-primary ideal I_2 of R_2. Then it is clear that $I_1' = I_1 \times R_2$ and $I_2' = R_1 \times I_2$ are $\delta_\times$-primary ideals of R by Theorem 4.1. Hence $I_1' \cap I_2' = I_1 \times I_2 = J$ is a 2-absorbing $\delta_\times$-primary ideal of R by Theorem 2.20.

(3)⇒(1) It is clear. □

Note that $I_1 \neq 0$ and $I_2 \neq 0$ in the hypothesis of Theorem 4.2. In the following result, it is possible that $I_1 = 0$ or $I_2 = 0$.

Theorem 4.3. *Let R_1 and R_2 be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$, and δ_1, δ_2 be expansion functions of ideals of R_1, R_2, respectively. Let $J = I_1 \times I_2$ be a proper ideal of R, for some ideals I_1, I_2 of R_1, R_2, respectively, such that for every $i \in \{1, 2\}$, if $I_i \neq R_i$, then $\delta_i(I_i) \neq R_i$. Then the following statements are equivalent.*
(1) *$J = I_1 \times I_2$ is a 2-absorbing $\delta_\times$-primary ideal of R.*
(2) *$I_1 = R_1$ and I_2 is a 2-absorbing δ_2-primary ideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing δ_1-primary ideal of R_1 or I_1 is a δ_1-primary ideal of R_1 and I_2 is a δ_2-primary ideal R_2.*

Proof. (1)⇒(2) Assume that $I_1 \times I_2$ is a 2-absorbing $\delta_\times$-primary ideal of R. If $I_1 = R_1$ ($I_2 = R_2$), then I_2 is a 2-absorbing δ_2-primary ideal of R_2 (I_1 is a 2-absorbing δ_1-primary ideal of R_1) by Theorem 4.1. So we may assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Hence, by hypothesis, $\delta_1(I_1) \neq R_1$ and $\delta_2(I_2) \neq R_2$. Suppose that I_1 is not a primary ideal of R_1. Then there are $a, b \in R_1$ such that $ab \in I_1$ but neither $a \in I_1$ nor $b \in \delta_1(I_1)$. Let $x = (a, 1), y = (1, 0)$, and $c = (b, 1)$. Then $xyc = (ab, 0) \in J$ but neither $xy = (a, 0) \in J$ nor $xc = (ab, 1) \in \delta_\times(J)$ nor $yc = (b, 0) \in \delta_\times(J)$, which is a contradiction. Thus I_1 is a primary ideal of R_1. Suppose that I_2 is not a primary ideal of R_2. Then there are $d, e \in R_2$ such that $de \in I_2$ but neither $d \in I_2$ nor $e \in \delta_2(I_2)$. Let $x = (1, d), y = (0, 1)$, and $c = (1, e)$. Then $xyc = (0, de) \in J$ but neither $xy = (0, d) \in J$ nor $xc = (1, de) \in \delta_\times(J)$ nor $yc = (0, e) \in \delta_\times(J)$, which is a contradiction. Thus I_2 is a primary ideal of R_2.

(2)⇒(1) If $J = I_1 \times R_2$ for some 2-absorbing δ_1-primary ideal I_1 of R_1 or $J = R_1 \times I_2$ for some 2-absorbing δ_2-primary ideal I_2 of R_2, then J is a 2-absorbing $\delta_\times$-primary ideal of R by Theorem 4.1. Hence assume that $J = I_1 \times I_2$ for some δ_1-primary ideal I_1 of R_1 and some δ_2-primary ideal I_2 of R_2. Then it is clear that $I_1' = I_1 \times R_2$ and $I_2' = R_1 \times I_2$ are $\delta_\times$-primary ideals of R by Theorem 4.1. Hence $I_1' \cap I_2' = I_1 \times I_2 = J$ is a 2-absorbing $\delta_\times$-primary ideal of R by Theorem 2.20. □

Theorem 4.4. *Let R_1 and R_2 be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$, and δ_1, δ_2 be expansion functions of ideals of R_1, R_2, respectively. Let $I = I_1 \times I_2$ be a nonzero proper ideal of R for some ideals I_1, I_2 of R_1, R_2, respectively, such that for every $i \in \{1, 2\}$, if $I_i \neq R_i$, then $\delta_i(I_i) \neq R_i$. Then I is a weakly 2-absorbing $\delta_\times$-primary ideal of R that is not 2-absorbing $\delta_\times$-primary if and only if one of the following conditions holds.*

(1) *$I = I_1 \times I_2$, where $I_1 \neq R_1$ is a nonzero weakly δ_1-primary ideal of R_1 that is not δ_1-primary and $I_2 = 0$ is a δ_2-primary ideal of R_2.*

(2) *$I = I_1 \times I_2$, where $I_2 \neq R_2$ is a nonzero weakly δ_2-primary ideal of R_2 that is not δ_2-primary and $I_1 = 0$ is a δ_1-primary ideal of R_1.*

Proof. Suppose that I is a nonzero weakly 2-absorbing $\delta_\times$-primary ideal of R that is not 2-absorbing $\delta_\times$-primary ideal. Then $I = I_1 \times I_2$ for some ideals I_1, I_2 of R_1 and R_2, respectively. Assume that $I_1 \neq 0$ and $I_2 \neq 0$. Then I is a 2-absorbing $\delta_\times$-primary ideal of R by Theorem 4.2, a contradiction. Therefore $I_1 = 0$ or $I_2 = 0$. Without loss of generality we may assume that $I_2 = 0$. We show that $I_2 = 0$ is a δ_2-primary ideal of R_2. Let $a, b \in R_2$ such that $ab \in I_2$, and let $0 \neq x \in I_1$. Since $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \in I$ and $(1, ab) \notin \delta_\times(I)$, we obtain $(x, a) = (x, 1)(1, a) \in I$ or $(x, b) = (x, 1)(1, b) \in \delta_\times(I)$, and so $a \in I_2$ or $b \in \delta_2(I_2)$. Thus $I_2 = 0$ is a δ_2-primary ideal of R_2. Next, we show that I_1 is a weakly δ_1-primary ideal of R_1. Observe that $I_1 \neq R_1$. For if $I_1 = R_1$, then it is clear that $R_1 \times 0$ is a 2-absorbing $\delta_\times$-primary ideal of R. Let $0 \neq ab \in I_1$, for some $a, b \in R_1$. Since $0 \neq (a, 1)(b, 1)(1, 0) \in I_1 \times 0$ and $(ab, 1) \notin I_1 \times 0$, we conclude $(a, 0) = (a, 1)(1, 0) \in \delta_\times(I_1 \times 0) = \delta_\times(I)$ or $(b, 0) = (b, 1)(1, 0) \in \delta_\times(I_1 \times 0) = \delta_\times(I)$. Thus $a \in I_1$ or $b \in \delta_1(I_1)$, and therefore I_1 is a weakly δ_1-primary ideal of R_1. Now, we show that I_1 is not δ_1-primary. Suppose that I_1 is a δ_1-primary ideal of R_1. Since $I_2 = 0$ is a δ_2-primary ideal of R_2, we conclude that $I = I_1 \times I_2$ is a 2-absorbing $\delta_\times$-primary ideal of R, a contradiction. Thus I_1 is a weakly δ_1-primary ideal of R_1 that is not δ_1-primary.

Conversely, suppose that (1) holds. Assume that $(0, 0) \neq (a, a')(b, b')(c, c') \in I = I_1 \times 0$. Since $a'b'c' = 0$ and $(0, 0) \neq (a, a')(b, b')(c, c') \in I_1 \times 0$, we conclude that $abc \neq 0$. Assume $(a, a')(b, b') \notin I$. We consider three cases.

Case one: Suppose that $ab \notin I_1$, but $a'b' = 0$. Since I_1 is a weakly δ_1-primary ideal of R_1, we have $c \in \delta_1(I_1)$. Since $I_2 = 0$ is a δ_2-primary ideal of R_2, we have $a' = 0$ or $b' \in \delta_2(I_2)$. Thus $(a, a')(c, c') \in \delta_\times(I)$ or $(b, b')(c, c') \in \delta_\times(I)$.

Case two: Suppose that $ab \notin I_1$ and $a'b' \neq 0$. Then $(c, c') \in \delta_1(I_1) \times \delta_2(I_2) = \delta_\times(I)$. Thus $(a, a')(c, c') \in \delta_\times(I)$ or $(b, b')(c, c') \in \delta_\times(I)$.

Case three: Suppose that $ab \in I_1$, but $a'b' \neq 0$. Since $0 \neq ab \in I_1$ and I_1 is a weakly δ_1- primary ideal of R_1, we have $a \in I_1$ or $b \in \delta_1(I_1)$. Since $a'b' \neq 0$ and $I_2 = 0$ is a δ_2-primary ideal of R_2, we have $c' \in \delta_2(I_2)$. Thus $(a, a')(c, c') \in \delta_\times(I)$ or $(b, b')(c, c') \in \delta_\times(I)$. Hence I is a weakly 2-absorbing primary ideal of R. Since I_1 is not a δ_1-primary ideal of R_1, I is not a 2-absorbing $\delta_\times$-primary ideal of R by Theorem 4.3. □

Theorem 4.5. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $2 \le n < \infty$, and $R_1, R_2, \ldots, R_n$ are commutative rings with $1 \neq 0$. Let $\delta_1, \ldots, \delta_n$ be expansion functions of ideals of $R_1, \ldots, R_n$, respectively. Let $J = I_1 \times \cdots \times I_n$ be a proper ideal of R, for some ideals $I_1, \ldots, I_n$ of $R_1, \ldots, R_n$, respectively, such that for every $i \in \{1, \ldots, n\}$, if $I_i \neq R_i$, then $\delta_i(I_i) \neq R_i$. Then the following statements are equivalent.*

(1) *J is a 2-absorbing $\delta_\times$-primary ideal of R.*

(2) *Either $J = \times_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, \ldots, n\}$, I_k is a 2-absorbing δ_k-primary ideal of R_k, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k\}$ or $J = \times_{t=1}^{n} I_t$ such that for some $k, m \in \{1, 2, \ldots, n\}$, I_k is a δ_k-primary ideal of R_k, I_m is a δ_m-primary ideal of R_m, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k, m\}$.*

Proof. We use induction on n. Assume that $n = 2$. Then the result is valid by Theorem 4.3. Thus let $3 \le n < \infty$ and assume that the result is valid when $H = R_1 \times \cdots \times R_{n-1}$. We prove the result when $R = H \times R_n$. Note that $\delta_H(I_1 \times \cdots \times I_{n-1} = \delta_1(I_1) \times \cdots \times \delta_n(I_n)$. By Theorem 4.3, J is a 2-absorbing $\delta_\times$-primary ideal of R if and only if either $J = L \times R_n$ for some 2-absorbing δ_H-primary ideal L of H or $J = H \times L_n$ for some 2-absorbing δ_n-primary ideal L_n of R_n or $J = L \times L_n$ for some δ_H-primary ideal L of H and some δ_n-primary ideal L_n of R_n. Since $I_i \neq R_1$ implies $\delta_i(I_i) \neq R_i$ by hypothesis, it should be clear that that a proper ideal Q of H is a δ_H-primary ideal of H if and only if $Q = \times_{t=1}^{n-1} I_t$ such that for some $k \in \{1, 2, \ldots, n-1\}$, I_k is a δ_k-primary ideal of R_k, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n-1\} \setminus \{k\}$. Thus the claim is now verified. □

Theorem 4.6. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $3 \le n < \infty$, and $R_1, R_2, \ldots, R_n$ are commutative rings with $1 \neq 0$. Let $\delta_1, \ldots, \delta_n$ be expansion functions of ideals of $R_1, \ldots, R_n$, respectively. Let $J = I_1 \times \cdots \times I_n$ be a nonzero proper ideal of R, for some ideals $I_1, \ldots, I_n$ of $R_1, \ldots, R_n$, respectively, such that for every $i \in \{1, \ldots, n\}$, if $I_i \neq R_i$, then $\delta_i(I_i) \neq R_i$. Then the following statements are equivalent.*

(1) *J is a weakly 2-absorbing $\delta_\times$-primary ideal of R.*

(2) *J is a 2-absorbing $\delta_\times$-primary ideal of R.*

(3) *Either $J = \times_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, \ldots, n\}$, I_k is a 2-absorbing δ_k-primary ideal of R_k, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k\}$ or $J = \times_{t=1}^{n} I_t$ such that for some $k, m \in \{1, 2, \ldots, n\}$, I_k is a δ_k-primary ideal of R_k, I_m is a δ_m-primary ideal of R_m, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k, m\}$.*

Proof. (1)$\Leftrightarrow$ (2). Since J is a proper ideal of R, we have $J = I_1 \times \cdots \times I_n$, where every I_i is an ideal of R_i, and $I_j \neq R_j$ for some $j \in \{1, \ldots, n\}$. Suppose that $J = I_1 \times I_2 \times \cdots \times I_n \neq (0, \ldots, 0)$ is a weakly 2-absorbing $\delta_\times$-primary ideal of R. Then there is an element $(0, \ldots, 0) \neq (a_1, a_2, \ldots, a_n) \in J$. Hence $(0, \ldots, 0) \neq (a_1, a_2, \ldots, a_n) = (a_1, 1, 1, \ldots, 1)(1, a_2, 1, \ldots, 1) \cdots (1, 1, \ldots, a_n) \in J$ implies there is a $j \in \{1, \ldots, n\}$ such that $b_j = 1$ and $(b_1, \ldots, b_n) \in \delta_\times(J) = \delta_1(I_1) \times \cdots \times \delta_n(I_n)$, where $b_1, \ldots, b_n \in \{1, a_1, \ldots, a_n\}$. Hence $\delta_j(I_j) = R_j$, and so $I_j = R_j$. Thus $J^3 \neq (0, \ldots, 0)$, and hence by Theorem 2.9(2), J is a 2-absorbing $\delta_\times$-primary ideal. The converse is obvious.

(2)$\Leftrightarrow$ (3). It is clear by Theorem 4.5. □

Recall that a ring R is called a von-Neumann regular ring if for every $x \in R$ there exists a $y \in R$ such that $xyx = x$. It is well-known that a von-Neumann regular ring is reduced.

Theorem 4.7. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $3 \le n < \infty$, and $R_1, R_2, \ldots, R_n$ are commutative rings with $1 \neq 0$. Let $\delta_1, \ldots, \delta_n$ be expansion functions of ideals of $R_1, \ldots, R_n$, respectively. Suppose that for every $i \in \{1, \ldots, n\}$, if whenever I_i is a proper ideal of R_i, then $\delta_i(I_i) \neq R_i$. Then the following statements are equivalent.*

(1) *Every proper ideal of R is a weakly 2-absorbing $\delta_\times$-primary ideal of R.*

(2) *$n = 3$ and R_i is a field for every $i \in \{1, 2, 3\}$.*

(3) *R is a von-Neumann regular ring with exactly three distinct maximal ideals.*

Proof. (1)$\Rightarrow$ (2). Suppose that every proper ideal of R is a weakly 2-absorbing $\delta_\times$-primary ideal of R. Assume $n \ge 4$. Then $J = 0 \times 0 \times 0 \times R_4 \times \cdots \times R_n$ is a proper ideal of R that is not a weakly 2-absorbing $\delta_\times$-primary ideal of R by Theorem 4.6. Hence $n = 3$. Without loss of generality, we may assume that R_1 is not a field. Then R_1 has a nonzero proper ideal I. Thus $J = I \times 0 \times 0$ is not a weakly 2-absorbing $\delta_\times$-primary ideal of R by Theorem 4.6. Hence R_i is a field for every $i \in \{1, 2, 3\}$.

(2)⇒ (3). No comments (it is clear).

(3)⇒ (1). Since every von-Neumann regular ring is reduced and every prime ideal of R is maximal, we conclude that R is ring-isomorphic to $R_1 \times R_2 \times R_3$, where each R_i is a field for every $i \in \{1, 2, 3\}$. If each R_i is a field for every $i \in \{1, 2, 3\}$, then it is easily verified (see Theorem 4.6) that every proper ideal of R is a weakly 2-absorbing $\delta_\times$-primary ideal of R. □

In view of Theorem 4.7, we have the following result.

Corollary 4.8. *Let R be a finite commutative ring with n prime ideals (maximal ideals), where $3 \leq n < \infty$), i.e., R is ring-isomorphic to $R_1 \times \cdots \times R_n$, where each R_i is a finite quasi-local ring, where $1 \leq i \leq n$. Let $\delta_1, \ldots, \delta_n$ be expansion functions of ideals of $R_1, \ldots, R_n$, respectively. Suppose that for every $i \in \{1, \ldots, n\}$, if whenever I_i is a proper ideal of R_i, then $\delta_i(I_i) \neq R_i$. Then the following statements are equivalent.*

(1) *Every proper ideal of R is a weakly 2-absorbing $\delta_\times$-primary ideal of R.*

(2) *$n = 3$ and R_i is a field for every $i \in \{1, 2, 3\}$.*

(3) *R is a von-Neumann regular ring with exactly three distinct prime ideals.*

(4) *R is a reduced ring with exactly three distinct prime ideals.*

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